education equals

Calculator Free
The Natural Logarithm and Anti-Differentiation

Time: 45 minutes
Total Marks: 45
Your Score: / 45

Question One: [2, 3, 3 = 8 marks] **CF**

Determine each of the following anti-derivatives, simplifying your answer where possible:

(a) $\int \frac{2}{2x-1}dx$

(b) $\int \frac{\sin x}{\cos x}dx$

(c) $\int \frac{e^x}{e^x-2}dx$

Question Two: **[4, 4 = 8 marks]** **CF**

Calculate each of the following definite integrals, simplifying your answers using logarithmic laws.

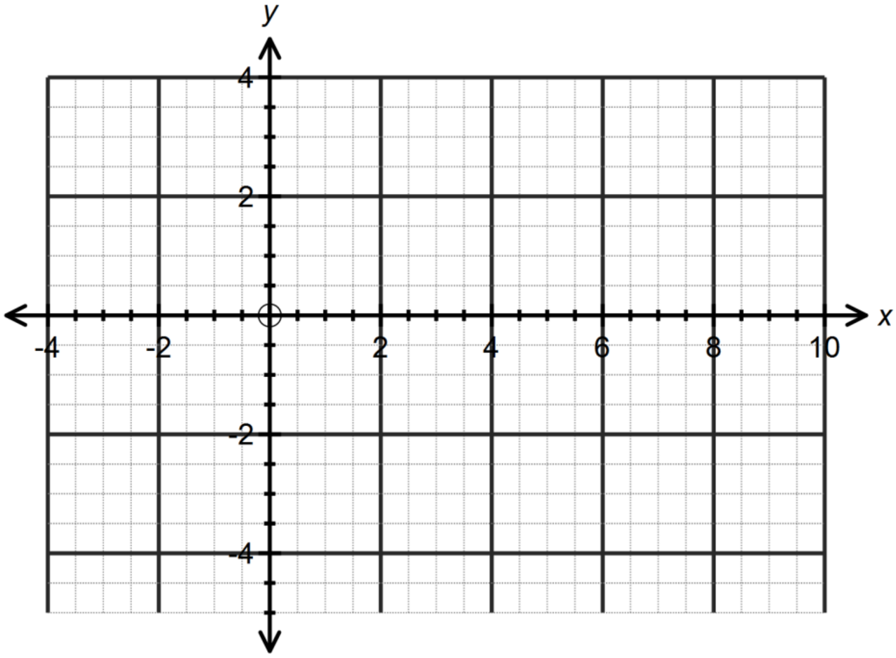
(a) $\int_{2}^{4} \frac{6}{3x-1}\, dx$

(b) $\int_{-1}^{5} \frac{x-1}{x^2-2x}\, dx$

Question Three: [3, 4, 4 = 11 marks] CF

Consider the function $f(x) = \frac{-1}{x-4} - 2$

(a) Sketch the function on the axes below.



(b) Calculate the area bounded by the function,the x – axis and the lines $x = 0$ and $x = 2$. Simplify your answer.

(c) Calculate the area bounded by the function, the y – axis and the lines $y=1$ and $y=4$.

Question Four: [4 marks] CF

Show that the exponential rule used to calculate compound interest, $A=P(1+r)^t$ can be written as $t=\dfrac{\ln A-\ln P}{\ln(1+r)}$.

Question Five: [3, 1, 1, 2, 2 =9 marks] CF

Newton's Law of Cooling allows us to monitor the rate at which the difference between the temperature of a body and its surrounds will cool over time.

This can be defined as: $\frac{d\theta}{dt} = k\theta$ where θ is the difference between the temperature of the body and the surrounding room temperature and t is the time in minutes since the body was introduced to the room.

In order to find a rule modelling θ in terms of t, we can first separate the variables as follows:

$$\frac{d\theta}{\theta} = k\ dt$$

We can then integrate both sides, as follows:

$$\int \frac{1}{\theta} d\theta = \int k\, dt$$

(a) Integrate and equate each side to show that $\ln\theta = kt + c$

A pizza is removed from a 200°C oven and put on the bench in a 25°C room. After 5 minutes, the temperature of the pizza is 120°C.

(b) Initially, what is the value of θ ?

(c) After 5 minutes, what is the value of θ ?

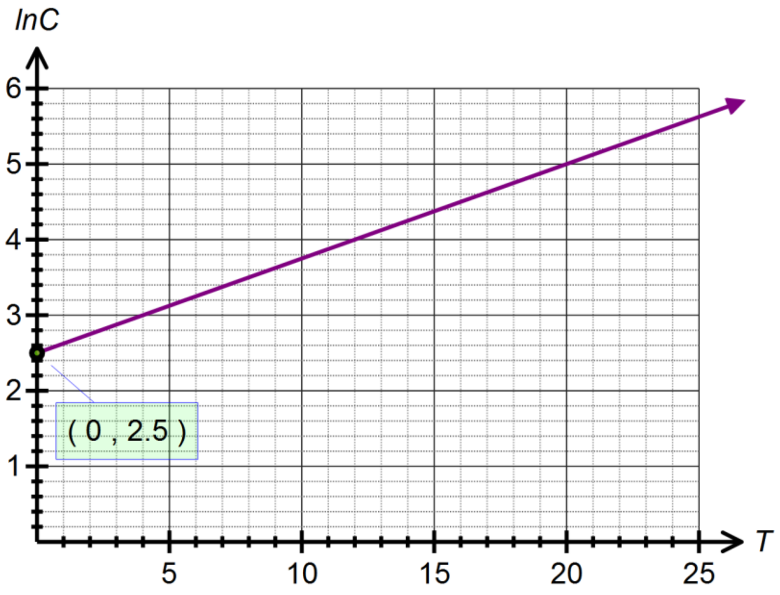
(d) Hence or otherwise determine the values of k and c.

(e) Hence determine when the pizza has reached room temperature.

Question Six: [3, 2 = 5 marks] CF

Synergy, the provider of electricity in Perth, monitor the maximum consumption of electricity over summer measured against the maximum temperatures.

Graphing the data provides us with the following graph, where C is maximum consumption in megawatts and T is the maximum temperature in degrees Celsius.



(a) Determine the equation of $\ln C$ in terms of T .

(b) Use your answer to (a) to determine the exponential function which models the energy consumption based on the maximum temperature recorded.

SOLUTIONS
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Question One: [2, 3, 3 = 8 marks] **CF**

Determine each of the following anti-derivatives, simplifying your answer where possible:

(a) $\int \frac{2}{2x-1}dx$

$= \ln|2x-1| + c$ ✓ ✓

(b) $\int \frac{\sin x}{\cos x}dx$

$= -\ln|\cos x| + c$ ✓ ✓ ✓

(c) $\int \frac{e^x}{e^x-2}dx$

$= \ln|e^x - 2| + c$ ✓ ✓ ✓

Question Two: **[4, 4 = 8 marks]** **CF**

Calculate each of the following definite integrals, simplifying your answers using logarithmic laws.

(a) $\int_2^4 \frac{6}{3x-1}\,dx$

$$= \left[2\ln|3x-1|\right]_2^4$$
$$= 2\ln|11| - 2\ln|5|$$
$$= 2(\ln 11 - \ln 5)$$
$$= 2\ln\left(\frac{11}{5}\right)$$

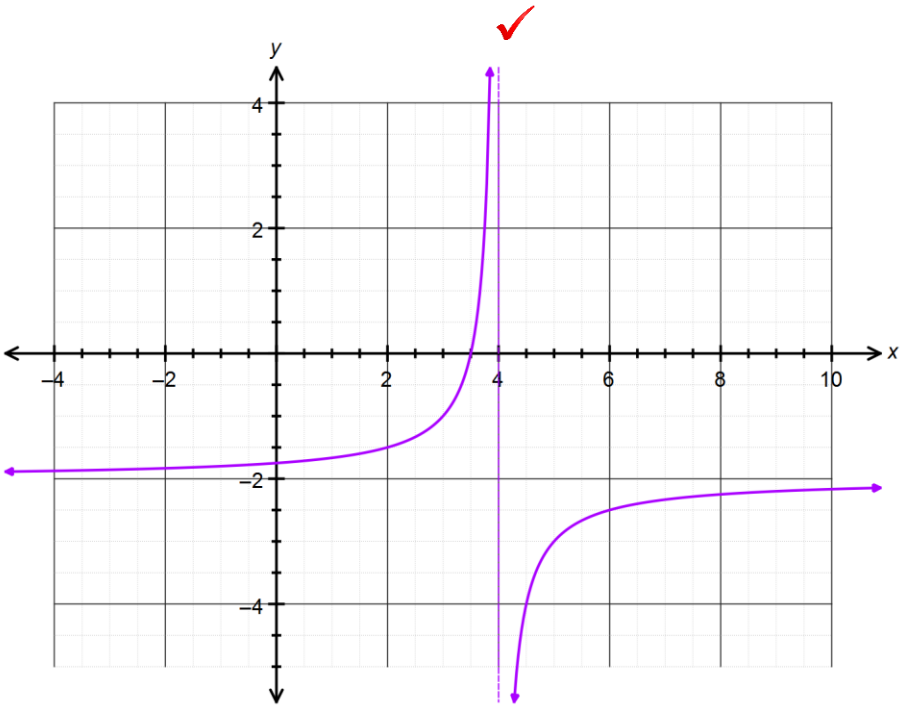
(b) $\int_{-1}^5 \frac{x-1}{x^2-2x}\,dx$

$$= \left[0.5\ln\left|x^2-2x\right|\right]_{-1}^5$$
$$= 0.5(\ln 15 - \ln 3)$$
$$= 0.5\ln 5$$

Question Three: **[3, 4, 4 = 11 marks]** **CF**

Consider the function $f(x)=\dfrac{-1}{x-4}-2$

(a) Sketch the function on the axes below.



(b) Calculate the area bounded by the function, the x – axis and the lines $x=0$ and $x=2$. Simplify your answer.

$$=\int_0^2 \frac{-1}{x-4}-2\ dx$$

$$=\left[-\ln|x-4|-2x\right]_0^2$$

$$=(-\ln 2-4)-(-\ln 4)$$

$$=\ln 4-\ln 2-4$$

$$=\ln 2-4$$

$$\therefore Area=|\ln 2-4|\,units^2$$

(c) Calculate the area bounded by the function, the y – axis and the lines $y=1$ and $y=4$.

$$x=\frac{-1}{y+2}+4$$

$$\int_1^4 \frac{-1}{y+2}+4 \quad dy$$

$$=\left[-\ln|y+2|+4x\right]_1^4$$

$$=(-\ln 6+16)-(-\ln 3+4)$$

$$=\ln 3-\ln 6+16-4$$

$$=\ln\frac{1}{2}+12 \; units^2$$

Question Four: [4 marks] CF

Show that the exponential rule used to calculate compound interest, $A=P(1+r)^t$ can be written as $t=\dfrac{\ln A-\ln P}{\ln(1+r)}$.

$$\ln A=\ln\left(P(1+r)^t\right)$$

$$\ln A=\ln P+t\ln(1+r)$$

$$t\ln(1+r)=\ln A-\ln P$$

$$t=\frac{\ln A-\ln P}{\ln(1+r)}$$

Question Five: [3, 1, 1, 2, 2 =9 marks] CF

Newton's Law of Cooling allows us to monitor the rate at which the difference between the temperature of a body and its surrounds will cool over time.

This can be defined as: $\frac{d\theta}{dt} = k\theta$ where θ is the difference between the temperature of the body and the surrounding room temperature and *t* is the time in minutes since the body was introduced to the room.

In order to find a rule modelling θ in terms of *t*, we can first separate the variables as follows:

$$\frac{d\theta}{\theta} = k\ dt$$

We can then integrate both sides, as follows:

$$\int \frac{1}{\theta} d\theta = \int k\, dt$$

(a) Integrate and equate each side to show that $\ln\theta = kt + c$

$$\int \frac{1}{\theta}\, d\theta = \ln\theta \quad ✓$$

$$\int k\ dt = kt + c \quad ✓✓$$

A pizza is removed from a 200°C oven and put on the bench in a 25°C room. After 5 minutes, the temperature of the pizza is 120°C.

(b) Initially, what is the value of θ ?

$\theta = 200 - 25 = 175^\circ C$ ✓

(c) After 5 minutes, what is the value of θ ?

$\theta = 120 - 25 = 95^\circ C$ ✓

(d) Hence or otherwise determine the values of k and c.

$$\ln 175 = c$$
$$\ln 95 = 5k + \ln 175$$
$$5k = \ln 95 - \ln 175$$
$$k = \frac{\ln 95 - \ln 175}{5}$$

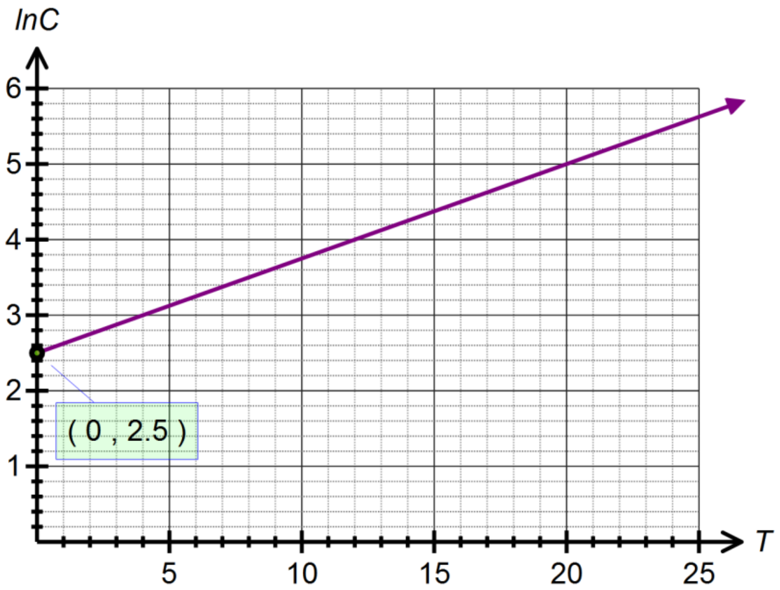
(e) Hence determine when the pizza has reached room temperature.

$$\ln 0 = \frac{\ln 95 - \ln 175}{5}t + \ln 175$$
$$1 - \ln 175 = \frac{\ln 95 - \ln 175}{5}t$$
$$t = \frac{5(1 - \ln 175)}{\ln 95 - \ln 175}$$

Question Six: **[3, 2 = 5 marks]** **CF**

Synergy, the provider of electricity in Perth, monitor the maximum consumption of electricity over summer measured against the maximum temperatures.

Graphing the data provides us with the following graph, where C is maximum consumption in megawatts and T is the maximum temperature in degrees Celsius.



(a) Determine the equation of $\ln C$ in terms of T .

$$m = \frac{0.5}{4} = \frac{1}{8}$$ ✓✓

$$\ln C = \frac{T}{8} + 2.5$$ ✓

(b) Use your answer to (a) to determine the exponential function which models the energy consumption based on the maximum temperature recorded.

$$\ln C = \frac{T}{8} + 2.5$$

$$C = e^{\frac{T}{8}+2.5}$$ ✓✓